

On electromagnetism and generalized energy-momentum tensor of the electromagnetic field in spaces with Finsler geometry

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December 10, 2010

Abstract

By using variational calculus and exterior derivative formalism, we proposed in [30] and [11] a new geometric approach to electromagnetism in pseudo-Finsler spaces. In the present paper, we provide more details, especially regarding generalized currents, the domain of integration and gauge invariance. Also, for flat pseudo-Finsler spaces, we define a generalized energy-momentum tensor consisting of two blocks, as the symmetrized Noether current corresponding to the invariance of the field Lagrangian with respect to spacetime translations. In curved spaces, one of the blocks of the generalized energy-momentum tensor is obtained by varying the field Lagrangian with respect to the metric tensor and the other one, by varying the same Lagrangian with respect to the nonlinear connection.

1 Introduction

Classical electromagnetism is one of the most "rounded" theories of physics and one has strong reasons to say that there is little to be added to it. There exist several beautiful geometrical descriptions of this theory in classical general relativity.

Still, what if spacetime is described not by Riemannian, but by Finslerian geometry? As G. Yu Bogoslovski and H. Goenner noticed, "spacetime may be not only in a state which is described by Riemann geometry but also in states which are described by Finsler geometry". A lot of authors have already considered Finslerian models for spacetime (see, for instance, [1], [4]-[7], [8]-[10], [15], [14], [20], [22], [23], [27], [28], [33], in order to cite just a few of them).

Regarding electromagnetism, we can expect that in spaces with Finslerian geometry, the corresponding equations would change and that we might even have to deal with some new quantities. In this paper, we will investigate from a mathematical point of view these possible changes.

The first idea we must have in mind is that in spaces with Finsler geometry, the metric tensor depends on the directional variables. Since Maxwell equations involve the metric tensor, we notice that the solutions (hence, the electromagnetic tensor) may also depend on these. This means that the natural space to work on is not the spacetime manifold M , but its tangent bundle TM .

Thus, we extend the classical ideas of electromagnetic field theory to Finsler spaces, as an application of the geometry of the tangent bundle TM .

Though we will speak throughout the chapter about Finsler spaces, all the theory remains valid, with minimal changes, for more general anisotropic spaces (Lagrange, generalized Lagrange ones).

This paper is a continuation of [11] and [30]. It is based on variational calculus and classical methods in theoretical physics (adapted to the tangent bundle). This approach offers an alternative to the existing one by R. Miron and collaborators, [17], [18], [15].

2 A brief overview of the Riemannian case

There are multiple definitions of the electromagnetic tensor, points of view and formulations of the basic equations of electromagnetism on Riemannian manifolds. Namely, the electromagnetic tensor can be regarded as the curvature of a line bundle over the given manifold or it can be described in terms of nonlinear/linear on TM as in [18], or in terms of differential forms.

In the following, we will adopt the language of differential forms, as it is the most tightly related to variational calculus.

Let us consider a pseudo-Riemannian manifold (M, g) of dimension 4, thought of as spacetime manifold. We denote local coordinates on M by $x = (x^i)_{i=\overline{0,3}}$ and use the numbering from 0 to 3. The first coordinate is regarded as the time coordinate and $(x^\alpha)_{\alpha=\overline{1,3}}$, as spatial coordinates. As required by general relativity, the metric $g = g(x)$ is supposed to have Lorentz signature $(+, -, -, -)$. Here are some other notations and conventions we will use in the following:

- Latin indices i, j, k, \dots take values from 0 to 3; Greek indices $\alpha, \beta, \gamma, \dots$ take values from 1 to 3;

- for a vector field $v = (v^i)_{i=\overline{0,3}}$ on M , \mathbf{v} will denote the spatial vector $\mathbf{v} = (v^\alpha)_{\alpha=\overline{1,3}}$.

- $_{,k}$ - partial derivative with respect to $\frac{\partial}{\partial x^k}$;

- $_{|k}$ - Levi-Civita covariant derivative with respect to $\frac{\partial}{\partial x^k}$; γ^i_{jk} - Christoffel symbols of g ;

- $g = \det(g_{ij})$; whenever it is not clear from the context whether we refer to g as the metric tensor or to the determinant of the corresponding matrix, we will specify this;

- $\flat : TM \rightarrow T^*M$, $\sharp : T^*M \rightarrow TM$ - musical isomorphisms (lowering/raising indices);

- d - exterior derivative of differential forms, $*$ - Hodge dual;

- $\epsilon_{i_1, \dots, i_p}$ - signature of the permutation (i_1, \dots, i_p) ;
- ∇ - gradient taken with respect to the spatial coordinates (x^α) ;
- $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, $d^3x = dx^1 \wedge dx^2 \wedge dx^3$;
- $d\Omega = \sqrt{|g|}d^4x$, $dV = \frac{\sqrt{|g|}}{\sqrt{g_{00}}}d^3x$ - the invariant Riemannian volume element on spacetime and on the spatial manifold respectively.

2.1 Distances, volumes, divergence, codifferential

Let us remind for the beginning some very quick facts about computation of time intervals, distances and spatial volumes in general relativity ([16], pp. 315-320).

The (squared) arclength element ds^2 on the spacetime manifold M can be written as

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta.$$

The spatial arclength element is defined as

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \quad \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}, \quad \alpha, \beta = \overline{1, 3}.$$

The determinant of the spacetime metric g is

$$g = -g_{00}\gamma;$$

we only consider reference frames for which both the determinant of the spatial metric γ and g_{00} are positive:

$$\gamma := \det(\gamma_{ij}) > 0 \text{ and } g_{00} > 0.$$

Here are some other relations we will use in the following.

The partial derivatives of $\sqrt{|g|}$ are given by:

$$\frac{\partial(\ln \sqrt{|g|})}{\partial x^j} = \frac{1}{2}g^{ih}g_{ih,j} = \gamma^i_{ji}.$$

Consequently, the divergence $\text{div}(V) = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (V^i \sqrt{|g|})$ of a vector field can be written in terms of covariant derivatives, as

$$\text{div}(V) = V^i_{|i}.$$

Also, by expressing g_{ij} as $\frac{1}{g^{-1}} \frac{\partial g^{-1}}{\partial g^{ij}}$, we get

$$d(\ln \sqrt{|g|}) = -\frac{1}{2}g_{ij}dg^{ij}. \quad (1)$$

The latter equality is particularly useful when varying Lagrangians with respect to the metric.

The codifferential of $\delta\xi = (-1)^p *^{-1} d*$ of a p -form $\xi = \frac{1}{p!} \xi_{i_1 i_2 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$ is a $(p-1)$ -form, with the property $\langle \eta, \delta\xi \rangle = \langle d\eta, \xi \rangle$, where $\langle \ , \ \rangle$ denotes the inner product of p -forms¹. For a 2-form, it is given by

$$(\delta\xi)^i = \xi^{ij}_{|j}.$$

2.2 4-potential and electromagnetic tensor

The 4-potential is geometrically described in classical general relativity as a 1-form

$$A = A_i(x) dx^i. \quad (2)$$

The electromagnetic tensor (or *Faraday 2-form*) is described as the 2-form

$$F = dA. \quad (3)$$

In local coordinates, this is

$$F = \frac{1}{2} F_{jk} dx^j \wedge dx^k, \quad (4)$$

where

$$F_{jk} = A_{k|j} - A_{j|k}. \quad (5)$$

Due to the symmetry of the Levi-Civita connection, the latter expression can be actually written in terms of partial derivatives only:

$$F_{jk} = A_{k,j} - A_{j,k}. \quad (6)$$

In the language of differential forms, the homogeneous Maxwell equations

$$F_{i[j|k} + F_{k|i]j} + F_{jk|i} = 0. \quad (7)$$

become

$$dF = 0. \quad (8)$$

Remark. There exist two possible approaches regarding the potential A and the electromagnetic tensor F .

1. One can consider as a fundamental object the electromagnetic tensor F , regarded as a closed 2-form on the manifold. In this case, the homogeneous Maxwell equation $dF = 0$, i.e., the closure condition for F , is taken as an axiom. If the manifold M is topologically "nice enough", then one can apply Poincaré's lemma, which entails the existence of a 1-form A , such that $F = dA$. That is, the existence of the 4-potential is seen as a consequence of the homogeneous Maxwell equations.

¹The inner product of two p -forms $\theta = \theta_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p}$ and $\psi = \psi_{j_1 \dots j_p} e^{j_1} \wedge \dots \wedge e^{j_p}$ is given by $\int g^{i_1 j_1} \dots g^{i_p j_p} \theta_{i_1 \dots i_p} \psi_{j_1 \dots j_p} d\Omega$, where the integral is taken on the whole manifold (assuming that the integrands have compact support).

2. Some authors consider the potential 1-form A as a fundamental object and *define* F as its exterior differential; in this approach, the homogeneous Maxwell equation is obtained as an identity.

Actually, for a Lagrangian theory of electromagnetism, it is essential to have both a 1-form A and a 2-form F , related by (8).

An important property of the electromagnetic field is *gauge invariance*. Namely, the field strength tensor F is invariant to transformations

$$A \mapsto A + d\psi,$$

where $\psi : M \rightarrow \mathbb{R}$ is a differentiable function.

2.3 Lagrangian, equations of motion and inhomogeneous Maxwell equations

The second pair of Maxwell equations (inhomogeneous Maxwell equations) and also, the equations of motion of charged particles in a given electromagnetic field are obtained by variational methods.

The *total action* attached to the field and to a system of particles is

$$S = - \sum mc \int ds - \sum \frac{q}{c} \int A_k(x) dx^k - \frac{1}{16\pi c} \int F_{ij} F^{ij} d\Omega. \quad (9)$$

Here, m, q, c are constants (m denotes the mass of a particle, q , its charge, c , the speed of light in vacuum), $d\Omega = \sqrt{|g|} d^4x$ is the invariant volume element on spacetime and the sums are taken over the particles in the system. The volume integral is taken over a bounded interval of time and over the whole spatial manifold, under the assumption that far away from sources, the field vanishes. Thus, we can actually think the integral as taken over a "large enough" compact domain in M .

The first term, $S_p := - \sum mc \int ds$, corresponds to the Lagrangian $L_p := - \sum mcds$ of the free particles.

The second term $S_{int} := - \sum \frac{q}{c} \int A_k(x) dx^k$, given by the Lagrangian $L_{int} := - \sum \frac{q}{c} A_k(x) dx^k$ characterizes the interaction between the particles and the field.

The third term $S_f := \frac{-1}{16\pi c} \int F_{ij} F^{ij} d\Omega = - \frac{1}{16\pi c} \int F * F d^4x$, characterizes the electromagnetic field in the given curved space.

By keeping the electromagnetic field fixed and varying the trajectory of a particle in the action S (which actually means varying the trajectory in $S_p + S_{int}$), one obtains the equations of motion of particles subject to both gravitational and electromagnetic field, i.e., the expression of the Lorentz force. If, conversely, in the action S we keep trajectories fixed and vary the electromagnetic field (which

actually means to vary the electromagnetic field in the sum $S_{int} + S_f$), we get the field equations, i.e., the second pair of Maxwell equations.

Let us notice that S_p and S_{int} are line integrals, while S_f is given by a volume integral. Hence, if we want to vary $S_{int} + S_f$, we have to write this sum as a single volume integral, too. This is achieved by means of the notion of charge density.

Charge density ρ is defined as the amount of electric charge in a given spatial volume and it is basically a function of time and spatial coordinates:

$$\rho = \rho(x).$$

The integral of ρ over a certain region of space provides the total charge situated inside that region:

$$q = \int \rho dV, \quad (10)$$

where $dV = \frac{\sqrt{|g|}}{\sqrt{g_{00}}} d^3x$ is the spatial volume element. In this writing under an integral, it is supposed that we actually see the charge distribution as "continuous". Total charge is invariant to coordinate changes.

For a discrete distribution of charges q_1, \dots, q_n in a given volume, we can still write the total charge in the form of the integral (10), if we define the charge density by means of the Dirac delta function:

$$\rho = \sum_{a=1}^n \frac{q_a}{\sqrt{\gamma}} \delta(\mathbf{x} - \mathbf{x}_{(a)}),$$

where $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{x}_{(a)}$ is the position vector of the charge q_a .

By using relation (10), S_{int} is written as

$$S_{int} = -\frac{1}{c} \int A_i J^i d\Omega,$$

where the quantities

$$J^i := \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0} \quad (11)$$

are the components of a vector field J , called the *4-current*.

Thus, the sum $S_1 := S_{int} + S_f$ can be written as a single integral as

$$S_1 = - \int \left(\frac{1}{c} A_i J^i + \frac{1}{16\pi c} F_{ij} F^{ij} \right) \sqrt{|g|} d^4x.$$

By varying the above Lagrangian with respect to the potential A , one gets the field equations, i.e., the *inhomogeneous Maxwell equation*:

$$\delta F = -\frac{4\pi}{c} J_b, \quad (12)$$

or, in local writing,

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \left(F^{ij} \sqrt{|g|} \right) = -\frac{4\pi}{c} J^i \iff F^{ij}{}_{|j} = -\frac{4\pi}{c} J^i. \quad (13)$$

Remark. The Maxwell equations and eventual choices of ψ in the transformations $A \mapsto A + d\psi$ do not completely determine the potential A . Hence, to A , one can still impose supplementary conditions (gauges). The most common is the *Lorenz gauge* $A^i{}_{|i} = 0$; under this condition the inhomogeneous Maxwell equations become:

$$-A^{i|j}{}_{|j} + A^j R^i{}_j = -\frac{4\pi}{c} J^i, \quad (14)$$

where $R^i{}_j = g^{ih} R^k{}_{hjk}$ are the components of the Ricci tensor.

By using inhomogeneous Maxwell equation, one obtains that the 4-current J identically satisfies the *continuity equation*:

$$\text{div}(J) = d(*J_b) = 0. \quad (15)$$

i.e.,

$$J^i{}_{|i} = 0. \quad (16)$$

From a physical point of view, the continuity equation is equivalent to the *charge conservation law*.

Let us now consider a single particle, subject to the action of a given (fixed) electromagnetic field and determine the trajectory of this particle. This can be achieved by varying the action S with respect to the trajectory. That is, we actually have to vary the action $S_2 := S_p + S_{int}$, written as a line intergral.

We notice that that the integral $S_p + S_{int}$ does not depend on the choice of the parameter on the path of integration. Thus, we can choose this parameter according to our wish. So, let us choose the arclength s as a parameter, With this choice, we have $\left\| \frac{dx}{ds} \right\| = \sqrt{g_{ij} \dot{x}^i \dot{x}^j} = \frac{ds}{ds} = 1$.

Thus, we can write

$$S_2 = -mc \int ds - \frac{q}{c} \int A_i(x) dx^i = - \int (mc \sqrt{g_{ij} \dot{x}^i \dot{x}^j} + A_i \dot{x}^i) ds, \quad (17)$$

where the integral is taken on some fixed compact interval $[s_0, s_1]$.

The Euler-Lagrange equations for the above Lagrangian are

$$\frac{D\dot{x}^i}{ds} = \frac{q}{c} F^i{}_j \dot{x}^j, \quad i = \overline{0, 3}, \quad (18)$$

where $\frac{D\dot{x}^i}{ds} = \frac{d\dot{x}^i}{ds} + \gamma^i{}_{jk} \dot{x}^j \dot{x}^k$ is the Levi-Civita covariant derivative.

The right hand sides of the above equations provide the expression of the *Lorentz force* in the given curved space. Also, the first two terms in (9) provide the canonical momentum

$$p_i = mc \frac{\dot{x}_i}{\|\dot{x}\|} + \frac{q}{c} A_i. \quad (19)$$

2.4 Energy-momentum tensor

Another important quantity in general relativity, is *energy-momentum tensor* (or *stress-energy tensor*) T . In classical general relativity, the energy-momentum tensor is symmetric and, for a closed system, its covariant divergence is zero.

A. In flat Minkowski space

In the Minkowski space (\mathbb{R}^4, η) ($\eta = \text{diag}(1, -1, -1, -1)$) of special relativity, it makes sense to speak about spacetime translations

$$x \mapsto x + a \quad (a - \text{constant 4-vector}).$$

The Lagrangians L_f, L_{int}, L_p in (9) are all invariant with respect to these translations (this can be easily checked, by noticing that neither of them depends explicitly on the spacetime coordinates x^i).

Moreover, in this case we have $g = \det(\eta_{ij}) = -1$ and covariant derivatives coincide with partial ones.

According to Noether's theorem, the invariance of an action

$$S = \frac{1}{c} \int \Lambda(q_{(l)}, \frac{\partial q_{(l)}}{\partial x^i}) d\Omega,$$

to translations implies that the quantities

$$\tilde{T}_i^k = q_{(l),i} \frac{\partial \Lambda}{\partial q_{(l),k}} - \delta_i^k \Lambda = 0$$

are conserved ($\text{div} \tilde{T} = 0$). They define a tensor of rank two (the *Noether current* attached to the Lagrangian).

The Noether current is generally not symmetric. Still, this situation can be "mended" by adding a divergence term:

$$T^{ik} = \tilde{T}^{ik} + \frac{\partial \psi^{ikl}}{\partial x^l},$$

(where $\psi^{ikl}(x) = -\psi^{ilk}(x)$ are functions of class at least two), which does not affect the value of the action integral (assuming, as usually, that on the boundary of the integration domain, the involved functions vanish). Thus, one obtains a symmetric tensor T of rank two, with

$$\frac{\partial T_i^k}{\partial x^k} = 0$$

The *energy-momentum tensor of the electromagnetic field* in flat space is defined as the symmetrized Noether current given by the invariance of the field Lagrangian L_f to spacetime translations.

For electromagnetism, we have $q_{(k)} = A_{(k)}$ and

$$\Lambda = -\frac{1}{16\pi}F_{ij}F^{ij}.$$

By supposing, at first, that $\rho = 0$ (which implies $J = 0$), we have: $\tilde{T}^l_i = \frac{1}{4\pi}(-F^{lk}A_{k,i} + \frac{1}{4}\delta^l_i F_{jk}F^{jk})$; by adding the quantity $\frac{1}{4\pi}A^i_{,l}F^{kl} = \frac{1}{4\pi}(A^i F^{kl})_{,l}$, one gets the energy-momentum tensor as:

$$T^l_i = \frac{1}{4\pi}(-F^{lk}F_{ik} + \frac{1}{4}\delta^l_i F_{jk}F^{jk}). \quad (20)$$

Thus, if $J = 0$, then

$$\text{div}(T) = 0.$$

In the situation when we have charged matter ($J \neq 0$), the energy-momentum tensor satisfies the identities:

$$T^j_{i,j} = -\frac{1}{c}F_{ij}J^j \quad (21)$$

(which can be proved by means of Maxwell equations). In brief,

$$\text{div}(T) = -\frac{1}{c}i_J F. \quad (22)$$

The quantity $\frac{1}{c}i_J F$ is called the *density of Lorentz force*.

B. In curved spaces

In general Relativity, the energy-momentum tensor T is defined by the relation

$$\delta_g S = \frac{1}{2c} \int T_{ik} \delta g^{ik} d\Omega = -\frac{1}{2c} \int T^{ik} \delta g_{ik} d\Omega. \quad (23)$$

Variation with respect to the metric leads to:

$$T_{ij} = \frac{1}{4\pi}(-F_j^k F_{ik} + \frac{1}{4}g_{ij}F_{lk}F^{lk}), \quad (24)$$

which agrees to the expression of the energy-momentum tensor in flat space.

Again, by using (both homogeneous and inhomogeneous) Maxwell equations, one gets that in curved pseudo-Riemannian spaces, the covariant divergence of the stress-energy tensor of the electromagnetic field is equal to minus the density of Lorentz force:

$$T^j_{i;j} = -\frac{1}{c}F_{ij}J^j. \quad (25)$$

Conclusion. In a geometric language, the above fundamental equations of electromagnetic field theory can be written briefly as:

- $F = dA$.
- $dF = 0$ – homogeneous Maxwell equation;
- $\delta F = -\frac{4\pi}{c}J_b$ – inhomogeneous Maxwell equation;
- $\text{div}(J) = 0$ – continuity equation;
- $\nabla \cdot T = -\frac{1}{c}i_J F$ – energy-momentum conservation (where $\nabla \cdot T$ denotes covariant divergence).

3 Some geometric structures in Finsler spaces

Let, again, M be a 4-dimensional differentiable manifold of class \mathcal{C}^∞ , thought of as spacetime manifold. This time we will also speak about the tangent bundle (TM, π, M) and denote $(x^i, y^i)_{i=\overline{0,3}}$ the coordinates in a local chart on TM ; we preserve the notations in the previous section, with the only difference that instead of Levi-Civita covariant derivatives, we will use other covariant derivation laws. Also, we denote partial derivation with respect to y^i by a dot: \cdot_i . We will sometimes call the base coordinates x^i positional variables and the fiber ones, directional variables.

A *Finsler fundamental function* on M , is a function $\mathcal{F} : TM \rightarrow \mathbb{R}$ with the properties, [25]:

1. $\mathcal{F} = \mathcal{F}(x, y)$ is smooth for $y \neq 0$;
2. \mathcal{F} is positive homogeneous of degree 1, i.e., $\mathcal{F}(x, \lambda y) = \lambda \mathcal{F}(x, y)$ for all $\lambda > 0$;
3. The *Finslerian metric tensor*:

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}, \quad (26)$$

is nondegenerate: $\det(g_{ij}(x, y)) \neq 0, \forall x \in M, y \in T_x M \setminus \{0\}$.

In the following, we will consider that the metric has signature $(+, -, -, -)$.

Remark. Strictly speaking, it would be more rigorous to preserve the term "Finslerian" for the case when the metric tensor g is positive definite and to call "pseudo-Finsler" the spaces in which the metric is only nondegenerate (of constant signature). But since a lot of authors already use in the latter case the term *Finsler*, we will also adopt this more relaxed terminology.

In a Finsler space, the element of arc length along a curve $t \mapsto x(t)$ is

$$ds = \mathcal{F}(x, \frac{dx}{dt})dt.$$

Finsler spaces are a generalization of pseudo-Riemannian manifolds, in which the coefficients of the metric tensor are no longer functions defined on M , but on the tangent bundle TM . Actually, if on a pseudo-Riemannian manifold, the tangent space at each point carries a pseudo-Euclidean metric structure, in a Finsler space, at each fixed point x^0 , the "norm" $\mathcal{F}(x_0, y)$ is generally, not given by a quadratic form. 1-homogeneity of \mathcal{F} in y insures that the integral $\int ds$ does not depend on eventual changes of the parameter along the curve (hence, the notion of arclength s is uniquely defined, no matter from which initial parameter t we start).

Given a (pseudo-)Finslerian metric tensor $g_{ij} = g_{ij}(x, y)$, the corresponding spatial metric is defined similarly to the pseudo-Riemannian case: $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}$, $\alpha, \beta \in \{1, 2, 3\}$ and its determinant is $\det(\gamma_{\alpha\beta}) = \frac{\sqrt{|g|}}{\sqrt{g_{00}}}$.

With respect to coordinate changes on the tangent bundle TM induced by coordinate changes $(x^i) \mapsto (\tilde{x}^i)$ on the base manifold M , i.e., under coordinate changes

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j \quad (27)$$

the quantities $\frac{\partial}{\partial y^i}$ have a tensorial rule of transformation: $\frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{y}^j}$, while the quantities $\frac{\partial}{\partial x^i}$ have a more complicated transformation law, [12], [15].

If we want to work with tensorial blocks only, then we have to use Ehresmann (nonlinear) connections on TM .

We will denote by $(N^{\bar{j}}_{\bar{i}})$ the coefficients of an Ehresmann connection, and by

$$\delta_i = \frac{\partial}{\partial x^i} - N^{\bar{i}}_i \frac{\partial}{\partial y^{\bar{i}}}, \quad \partial_{\bar{i}} = \frac{\partial}{\partial y^{\bar{i}}}, \quad (dx^{\bar{i}}, dy^{\bar{i}} = dy^i + N^{\bar{i}}_j dx^j) \quad (28)$$

the elements of the corresponding *adapted basis* and of its dual cobasis. Thus, with respect to coordinate changes (27), δ_i and $\partial_{\bar{i}}$ have tensorial rules of transformation, i.e., $\delta_i = \frac{\partial x^{j'}}{\partial x^i} \delta_{j'}$, $\partial_{\bar{i}} = \frac{\partial x^{j'}}{\partial x^i} \partial_{j'}$.

In the adapted basis, any vector field V on TM can be written as $V = V^i \delta_i + V^{\bar{i}} \partial_{\bar{i}}$; the component

$$hV = V^i \delta_i$$

is a vector field, called the *horizontal* component of V , while

$$vV = V^{\bar{i}} \partial_{\bar{i}}$$

is also a vector field, called its *vertical* component. Similarly, a 1-form ω on TM can be decomposed into invariant blocks as $\omega = \omega_i dx^i + \omega_{\bar{i}} \delta y^{\bar{i}}$, with

$$h\omega = \omega_i dx^i$$

called the *horizontal* component, and

$$v\omega = \omega_{\bar{i}} \delta y^{\bar{i}}$$

the *vertical* one, [15]. Accordingly, any tensor field on TM is decomposed with respect to the Ehresmann connection into invariant blocks.

In the following, whenever needed to make a clear distinction, we will denote by i, j, k, \dots indices corresponding to horizontal geometric objects and by $\bar{i}, \bar{j}, \bar{k}, \dots$ (with bars), indices corresponding to vertical ones - though, unless needed, we will not be too strict in this respect. By capital letters A, B, C, \dots we will always denote indices which take values corresponding to both distributions: $A, B, C, \dots \in \{i, j, k, \dots, \bar{i}, \bar{j}, \bar{k}, \dots\}$

Let us complete g up to a block metric (an *hv-metric*, [15]) on TM :

$$G_{AB}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + v_{\bar{i}\bar{j}}(x, y) \delta y^{\bar{i}} \otimes \delta y^{\bar{j}}. \quad (29)$$

where g is the given Finslerian metric tensor and v is a positive definite metric tensor². Thus, (TM, G) becomes a pseudo-Riemannian space and we can speak about the Riemannian (invariant) volume element on TM :

$$d\Omega = \sqrt{|G|} d^4x \wedge d^4y.$$

where $G = \det(G_{AB})$ (we have written d^4y instead of δ^4y in the above exterior product, since $d^4x \wedge d^4y = d^4x \wedge \delta^4y$). The determinant G is, obviously,

$$G = g \cdot v, \quad g = \det(g_{ij}), \quad v = \det(v_{\bar{i}\bar{j}}).$$

The volume element $d\Omega$ defines a volume element $d\Omega_M$ on M by:

$$d\Omega_M = \sigma(x) d^4x, \quad \sigma(x) = \int_{D_x} \sqrt{|G|} d^4x \wedge d^4y,$$

where $D_x = \{y \in T_x M \mid v_{\bar{i}\bar{j}}(x, y) y^{\bar{i}} y^{\bar{j}} \leq r^2\}$ and $r = \sqrt[4]{2/\pi^2}$ is chosen such that the 3-sphere of radius r in the 4-dimensional Euclidean space has the volume equal to 1. This volume element generalizes the idea of Holmes-Thompson volume in [24]³. Regarding integration with respect to x , we can assume that

²Assuming that the topological space M is metrizable, it appears as advantageous to choose, for instance, a metric v which provides the topology of M . In the case when (M, g) is the Minkowski space, the manifold topology of $M = \mathbb{R}^4$ is the Euclidean one, hence we can choose v as the Euclidean metric.

³The classical idea of Holmes-Thompson volume involves integration on the indicatrix $I_x = \{y \in T_x M \mid g_{ij} y^i y^j = 1\}$. If the Finsler metric g is not positive definite, the indicatrix I_x is generally non-compact, hence this classical idea cannot be applied in our case. Choosing as vertical part v of the metric G on TM a positive definite one (for instance, related to the spacetime topology) and integrating on balls given by the metric v solves this problem.

the corresponding domain is a "large enough" compact one (in the assumption that far away from sources, the field is negligible and the considered time interval is a bounded one).

Having a metric structure on TM , there now make sense notions such as: Hodge dual $*$ or codifferential δ of p -forms on TM , gradient of a function and divergence of a vector field.

The *divergence* of a vector field $V = V^i \delta_i + V^{\bar{i}} \partial_{\bar{i}}$ on TM is obtained from the relation $\mathcal{L}_V d\Omega = \text{div} V d\Omega$. In the adapted frame to an arbitrary nonlinear connection, the divergence of a vector field is expressed as

$$\text{div} V = \frac{1}{\sqrt{|G|}} \left[\delta_i (V^i \sqrt{|G|}) + \partial_{\bar{i}} (V^{\bar{i}} \sqrt{|G|}) \right] - N^{\bar{j}}_{i \cdot \bar{j}} V^i. \quad (30)$$

In particular, if the vertical block of the metric is a Riemannian one $v = v(x)$, then:

- the functions

$$N^{\bar{i}}_{j \cdot \bar{j}} = \gamma^{\bar{i}}_{j \cdot k}(x) y^k, \quad (31)$$

where $\gamma^{\bar{i}}_{j \cdot k}(x)$ are the Christoffel symbols of v , are the coefficients of a nonlinear connection on M ;

- in terms of this nonlinear connection, the expression of the divergence is simplified as:

$$\text{div} V = \frac{1}{\sqrt{|g|}} \left[\delta_i (V^i \sqrt{|g|}) + \partial_{\bar{i}} (V^{\bar{i}} \sqrt{|g|}) \right]. \quad (32)$$

The codifferential of any p -form $\xi = \frac{1}{p!} \xi_{i_1 i_2 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p}$ on TM is the $(p-1)$ -form $\delta \xi = (-1)^p *^{-1} d*$; it can be also calculated from the relation $\langle \eta, \delta \xi \rangle = \langle d\eta, \xi \rangle$, where \langle , \rangle denotes the inner product of p -forms⁴.

For a 2-form

$$\xi = \frac{1}{2} \xi_{ij} dx^i \wedge dx^j + \xi_{ia} dx^i \wedge \delta y^a + \frac{1}{2} \xi_{ab} \delta y^a \wedge \delta y^b$$

on TM , the codifferential of ξ is a 1-form $\delta \xi = \omega_i dx^i + \omega_a \delta y^a$ whose contravariant components are given by:

$$\begin{aligned} \omega^i &= \frac{1}{\sqrt{|G|}} [\delta_j (\xi^{ij} \sqrt{|G|}) + \partial_{\bar{j}} (\xi^{i\bar{j}} \sqrt{|G|})] - \xi^{ij} N^{\bar{k}}_{j \cdot \bar{k}}; \\ \omega^{\bar{i}} &= \frac{1}{\sqrt{|G|}} [\delta_j (\xi^{\bar{i}j} \sqrt{|G|}) + \partial_{\bar{j}} (\xi^{\bar{i}\bar{j}} \sqrt{|G|})] - \frac{1}{2} \xi^{jk} R^{\bar{i}}_{jk} - \xi^{\bar{i}j} N^{\bar{k}}_{j \cdot \bar{k}} + \xi^{\bar{k}j} N^{\bar{i}}_{j \cdot \bar{k}}. \end{aligned}$$

⁴The inner product of two p -forms $\theta = \theta_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p}$ and $\psi = \psi_{j_1 \dots j_p} e^{j_1} \wedge \dots \wedge e^{j_p}$ is traditionally given by $\int g^{i_1 j_1} \dots g^{i_p j_p} \theta_{i_1 \dots i_p} \psi_{j_1 \dots j_p} d\Omega$, where the integral is taken on the whole manifold (and it makes sense, for instance, for objects with compact support). In the case of TM , we will not integrate on the whole TM , but on a compact domain as specified above.

Choosing a nonlinear connection and a notion of covariant derivation or another can help to express locally in a more or less elegant form the obtained equations.

It appears as convenient to choose the following linear connection $D\Gamma(N)$, inspired from [15] (just - with a different covariant derivation law for vertical fields)⁵:

$$\begin{aligned} X^j_{|i} &= \delta_i X^j + L^j_{hi} X^h, & X^j_{\cdot\bar{i}} &= \frac{\partial X^j}{\partial y^{\bar{i}}}, \\ X^{\bar{j}}_{|i} &= \delta_i X^{\bar{j}} + L^{\bar{j}}_{\bar{h}i} X^{\bar{h}}, & X^{\bar{j}}_{\cdot\bar{i}} &= \frac{\partial X^{\bar{j}}}{\partial y^{\bar{i}}}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} L^i_{jk} &= \frac{1}{2} g^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}), \\ L^{\bar{i}}_{\bar{j}k} &= N^{\bar{i}}_{k\cdot\bar{j}} + \frac{1}{2} v^{\bar{i}\bar{h}} (\delta_k v_{\bar{h}\bar{j}} - N^{\bar{i}}_{k\cdot\bar{j}} v_{\bar{h}} - N^{\bar{i}}_{k\cdot\bar{h}} v_{\bar{j}}). \end{aligned} \quad (34)$$

The above linear connection is a *distinguished connection*, [15], meaning that it preserves the distributions generated by the Ehresmann connection N and it is h -metrical, i.e., $g_{ij|k} = 0$, $v_{\bar{i}\bar{j}|k} = 0$, $\forall i, j, k, \bar{i}, \bar{j} \in \{0, 3\}$. Its only nonvanishing components of its torsion tensor T are

$$\begin{aligned} R^{\bar{i}}_{jk} &= \delta y^{\bar{i}}(T(\delta_k, \delta_j)) = \delta_k N^{\bar{i}}_j - \delta_j N^{\bar{i}}_k; \\ P^{\bar{i}}_{j\bar{k}} &= \delta y^{\bar{i}}(T(\partial_{\bar{k}}, \delta_j)) = N^{\bar{i}}_{j\cdot\bar{k}} - L^{\bar{i}}_{\bar{k}j}. \end{aligned}$$

For the linear connection $D\Gamma(N)$ above defined, there hold the relations:

$$\delta_j(\ln \sqrt{|g|}) = L^i_{ji}, \quad \delta_j(\ln \sqrt{|v|}) - N^{\bar{i}}_{j\cdot\bar{k}} = -P^{\bar{i}}_{i\bar{j}}. \quad (35)$$

Consequently, the divergence of a horizontal vector field $V^H = V^i \delta_i$ on TM can be written as:

$$\text{div}(V^H) = (V^i_{|i} - P^{\bar{j}}_{i\bar{j}} V^i). \quad (36)$$

Another important notion for a Finsler space is the *Cartan tensor* C given by

$$C^i_{j\bar{k}} = \frac{1}{2} g^{ih} g_{hj\cdot\bar{k}}.$$

It also has the property that

$$\frac{\partial(\ln \sqrt{|g|})}{\partial y^{\bar{j}}} = C^i_{i\bar{j}}.$$

⁵The choice of this linear connection instead of the classical metrical linear connection, [15], appeared as a little bit more comfortable when expressing, for instance, the homogeneous Maxwell equation in coordinates. This is just an example. All the results can be re-expressed in terms of other linear connections.

Particular case: If v is a Riemannian metric, i.e., if

$$G_{AB}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + v_{\bar{i}\bar{j}}(x)\delta y^{\bar{i}} \otimes \delta y^{\bar{j}}. \quad (37)$$

and N is given by (31), then

$$L^{\bar{i}}_{\bar{j}k} = \gamma^{(v)i}_{jk} = N^{\bar{i}}_{k\bar{j}}$$

and the only nonvanishing components of the torsion remain $R^{\bar{i}}_{jk} = r_l^i{}_{jk}(x)y^l$, where $r_l^i{}_{jk}$ are the local components of the curvature of the Levi-Civita connection of v . Moreover, in this case,

$$\delta_j(\ln \sqrt{|v|}) = \gamma^{(v)i}_{ji}, \quad P^{\bar{i}}_{i\bar{j}} = 0.$$

This will simplify a lot of calculations. For instance, the divergence of a vector field $V = V^i\delta_i + V^{\bar{i}}\partial_{\bar{i}}$ and the codifferential $\omega = \delta\xi$ of a 2-form $\xi = \frac{1}{2}\xi_{ij}dx^i \wedge dx^j + \xi_{i\bar{j}}dx^i \wedge \delta y^{\bar{j}} + \frac{1}{2}\xi_{\bar{i}\bar{j}}\delta y^{\bar{i}} \wedge \delta y^{\bar{j}}$ on TM are given in terms of covariant derivatives (34):

$$\text{div}V = V^i{}_{|i} + V^{\bar{i}}{}_{\bar{i}} + V^{\bar{i}}C^j_{j\bar{i}},$$

and

$$\begin{aligned} \omega^i &= \xi^{ij}{}_{|j} + \xi^{i\bar{j}}{}_{\bar{j}} + \xi^{i\bar{j}}C^l_{l\bar{j}}; \\ \omega^{\bar{i}} &= \xi^{\bar{i}j}{}_{|j} + \xi^{\bar{i}\bar{j}}{}_{\bar{j}} + \xi^{\bar{i}\bar{j}}C^l_{l\bar{j}} - \frac{1}{2}\xi^{jk}R^{\bar{i}}_{jk}. \end{aligned}$$

4 4-potential 1-form

If we want to use variational calculus in order to provide a generalization of electromagnetic field theory to Finsler spaces, we need a generalization of the notion of 4-potential.

Let us now see how does the notion of 4-potential transform in the case when the geometry of the space is no longer Riemannian, but Finslerian, i.e., when

$$g_{ij} = g_{ij}(x, y).$$

We notice that the inhomogeneous Maxwell equations involve the components of the metric tensor, which depend on the fiber coordinates y^i . It becomes clear that generally, the solutions A would depend on both x and y . Also, the equations themselves could become more complicated.

Consequently, from now on, we will consider

$$A = A(x, y). \quad (38)$$

For reasons which will be clarified later, we will also assume that the components A_i are 0-homogeneous in y :

$$A_i(x, \lambda y) = A_i(x, y).$$

That is, we will allow A to depend on the direction of y , but not on its magnitude.

In the following, we will focus on the action (9) and determine the consequences of the y -dependence of the metric g - and of the potential A . In pseudo-Finslerian spaces, the first term S_p formally remains the same, with the only difference that in the expression $ds^2 = g_{ij}(x, y)y^i y^j dt$, g_{ij} depends on $y = \dot{x}$.

5 Faraday 2-form and homogeneous Maxwell equations

Let us define the generalized Faraday 2-form (the electromagnetic tensor) in the same way as in Riemannian spaces:

$$F = dA; \tag{39}$$

in local coordinates, this is

$$F := \frac{1}{2} F_{ij} dx^i \wedge dx^j + F_{i\bar{j}} dx^i \wedge \delta y^{\bar{j}}, \tag{40}$$

In terms of adapted derivatives, components of the Faraday 2-form are expressed as

$$F_{ij} = \delta_i A_j - \delta_j A_i, \quad F_{i\bar{j}} = -\partial_{\bar{j}} A_i$$

and in terms of covariant derivatives (33), we get

$$F_{ij} = A_{j|i} - A_{i|j}, \quad F_{i\bar{j}} = -A_{i.\bar{j}}. \tag{41}$$

In particular, if $A = A(x)$ does not depend on the directional variables, we get $F = \frac{1}{2}(A_{j|i} - A_{i|j})dx^i \wedge dx^j$, which is similar to the expression in [15], [18].

The electromagnetic tensor F remains invariant under transformations

$$A(x, y) \mapsto A(x, y) + d\lambda(x), \tag{42}$$

where $\lambda : M \rightarrow \mathbb{R}$ is a scalar function, since $d(A + d\lambda) = dA + d(d\lambda) = dA$.

Since F is, by definition, a closed 2-form, its exterior derivative identically vanishes. In other words:

Proposition 1 *There holds the generalized homogeneous Maxwell equation:*

$$dF = 0. \quad (43)$$

Obviously, in local coordinates, equation (43) will acquire different forms, depending on the chosen Ehresmann connection N and linear connection $D\Gamma(N)$.

In terms of covariant derivatives (34), equation (43) is read as:

$$\begin{aligned} F_{ij|k} + F_{ki|j} + F_{jk|i} &= - \sum_{(i,j,k)} R_{jk}^{\bar{h}} F_{i\bar{h}}; \\ F_{\bar{i}j|k} + F_{k\bar{i}|j} + F_{jk|\bar{i}} &= P_{j\bar{i}}^{\bar{h}} F_{k\bar{h}} - P_{k\bar{i}}^{\bar{h}} F_{j\bar{h}}, \quad F_{k\bar{i}|\bar{j}} + F_{j\bar{k}|\bar{i}} = 0. \end{aligned}$$

If $v = v(x)$ and $N_j^{\bar{i}}(x, y) = \gamma_{jk}^{(v)\bar{i}} y^k$, then the second set of equations becomes

$$F_{\bar{i}j|k} + F_{k\bar{i}|j} + F_{jk|\bar{i}} = 0.$$

The first set in the above is the analogue (in the nonholonomic frame $(\delta_i, \partial_{\bar{i}})$ on TM) of the usual homogeneous Maxwell equations. In the cases when we can choose the nonlinear connection N such that the horizontal distribution is integrable, then also the right hand sides of the first set of equations vanish.

In the above, we have started from A as an *a priori* given object and defined F as its exterior derivative. Let us now proceed conversely and suppose that F is given. As we have shown in ([30]), under the assumptions that: the manifold M is contractible and F is a closed 2-form with vanishing $\delta y^i \wedge \delta y^j$ component, i.e.,

$$F := \frac{1}{2} F_{ij} dx^i \wedge dx^j + F_{i\bar{j}} dx^i \wedge \delta y^{\bar{j}}, \quad dF = 0,$$

there exists a horizontal form A such that $F = dA$.

6 Inhomogeneous Maxwell equations

As we have seen, in Finsler spaces the 4-potential A and the generalized Faraday 2-form are defined on the tangent bundle TM .

The interaction term of the total action becomes

$$S_{int} = - \sum \frac{q}{c} \int A = - \sum \frac{q}{c} \int A_i(x, \dot{x}) dx^i.$$

In the classical Riemannian case, the above integral is transformed into one on a domain in the spacetime M . In our case, we will transform it into an integral on a domain in TM . That is, we must write total charge as an integral (on a domain as specified above):

$$q = \int \frac{\rho(x)}{\sqrt{g_{00}}} \sqrt{G} d^3x \wedge d^4y.$$

This way, S_{int} will be given by

$$-\frac{1}{c} \int A_i \frac{\rho(x)}{\sqrt{g_{00}}} \frac{dx^i}{dx^0} \sqrt{G} d^4x \wedge d^4y$$

With the notation

$$J^i = \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0}, \quad (44)$$

the integral $\int A_k dx^k$ is written as

$$-\frac{q}{c} \int A_k dx^k = -\frac{1}{c} \int A_i J^i d\Omega. \quad (45)$$

The above expression is formally similar to the one in the pseudo-Riemannian case, though, here, the volume element is considered on a certain domain in the tangent bundle TM .

The quantities J^i (interpreted as components of the 4-current) thus define the horizontal component of some vector field

$$J = J^i \delta_i + J^{\bar{i}} \partial_{\bar{i}}$$

on TM .

The field equations can be obtained by varying with respect to the potential A the action

$$S_1 = -\frac{1}{c} \int A - \frac{1}{16\pi c} \int F * F d^4x \wedge d^4y. \quad (46)$$

This variation leads to:

$$\frac{1}{\sqrt{G}} \{ (F^{ij} \sqrt{G})_{;j} - F^{ij} N_{j\bar{k}}^{\bar{k}} \sqrt{G} \} + \frac{1}{\sqrt{G}} (\tilde{F}^{i\bar{j}} \sqrt{G})_{;\bar{j}} = -\frac{4\pi}{c} J^i. \quad (47)$$

With $v = v(x)$ and $N_{\bar{j}}^{\bar{i}}(x, y) = \gamma_{\bar{j}k}^{(v)\bar{i}}(x) y^k$, we have

$$\frac{1}{\sqrt{|g|}} \{ \delta_j (F^{ij} \sqrt{|g|}) + (F^{i\bar{j}} \sqrt{|g|})_{;\bar{j}} \} = -\frac{4\pi}{c} J^i. \quad (48)$$

Notes: 1) In the integral above, in order to make sure that the expression has physical sense, we might need to adjust measurement units so as to have $[F_{ij}] = [F_{i\bar{j}}]$. This can be done, by considering the fiber coordinates $y^{\bar{i}}$ as having the same measurement units as the base ones (eventually, by multiplying them by a constant, [30]).

2) We remark a certain resemblance between the term $(F^{i\bar{j}}\sqrt{G})_{;\bar{j}}$ and the idea of bound current in a material medium.

Equations (48) gave the idea to formally generalize the *inhomogeneous Maxwell equation* as

$$\delta F = -\frac{4\pi}{c}J_b. \quad (49)$$

In local coordinates, this is:

$$\begin{aligned} F^{ij}_{|j} + F^{i\bar{j}}_{;\bar{j}} + Q^i &= -\frac{4\pi}{c}J^i, \\ F^{\bar{i}j}_{|j} + Q^{\bar{i}} &= -\frac{4\pi}{c}J^{\bar{i}}, \end{aligned} \quad (50)$$

where

$$\begin{aligned} Q^i &= -F^{ij}P_{j\bar{k}}^{\bar{k}} + F^{i\bar{j}}\partial_{\bar{j}}(\ln \sqrt{|G|}) \\ Q^{\bar{i}} &= -\frac{1}{2}F^{jk}R_{jk}^{\bar{i}} - F^{j\bar{k}}P_{j\bar{k}}^{\bar{i}} - F^{\bar{i}j}P_{j\bar{k}}^{\bar{k}}. \end{aligned} \quad (51)$$

In particular, if $v = v(x)$ and $N_{\bar{j}}^{\bar{i}}(x, y) = \gamma^{(v)\bar{i}}_{\bar{j}k}(x)y^k$, this yields

$$\begin{aligned} F^{ij}_{|j} + F^{i\bar{j}}_{;\bar{j}} + F^{i\bar{j}}C^l_{l\bar{j}} &= -\frac{4\pi}{c}J^i \\ F^{\bar{i}j}_{|j} - \frac{1}{2}F^{jk}R_{jk}^{\bar{i}} &= -\frac{4\pi}{c}J^{\bar{i}}, \end{aligned} \quad (52)$$

The first set of equations is nothing but (52) obtained by variational methods, while the second one is new. We notice the appearance of the quantities $J^{\bar{i}}$ (due to both the Finslerian character of the space and the nonholonomy of the frame we used) which are "coupled" on TM to the usual components of the 4-current J^i .

In the following, we will see that $J^{\bar{i}}$ play an important role in the continuity equation and in the Finslerian analogue of energy-momentum conservation law.

7 Continuity equation and gauge invariance

Above, we have seen that

$$-\frac{4\pi}{c}J_b = \delta F. \quad (53)$$

There immediately follows: $-\frac{4\pi}{c}\delta J_b = \delta\delta F = 0^6$, which is, $div(J) = 0$. In other words:

⁶We have used the identity $\delta\delta\omega = (-1)^{2p}(*^{-1}d*)(*^{-1}d*)\omega = *^{-1}dd\omega = 0$.

Proposition 2 *There holds the generalized continuity equation:*

$$\operatorname{div}(J) = 0. \quad (54)$$

We have seen above that the electromagnetic tensor F is invariant under transformations $A(x, y) \mapsto \tilde{A}(x, y) := A(x, y) + d\lambda(x)$ of the 4-potential. It means that, in the general action (9), the first term S_p and the third one S_f will also be invariant.

The continuity equation (54) insures that, with respect to the above transformations, $\tilde{S}_{int} = -\int \tilde{A}_i J^i \sqrt{|G|} d^4x \wedge d^4y$ equals S_{int} plus a boundary term.

Indeed, we have (omitting the minus sign in front of the integral):

$$\int \tilde{A}_i J^i d\Omega = \int (A_i + \frac{\partial \lambda}{\partial x^i} J^i) \sqrt{|G|} d^4x \wedge d^4y.$$

Since λ depends only on x , we can write $\frac{\partial \lambda}{\partial x^i} = \lambda_{;i}$, hence the term to be added to $\int A_i J^i \sqrt{|G|} d^4x \wedge d^4y$ is

$$\int \frac{\partial \lambda}{\partial x^i} J^i \sqrt{|G|} d^4x \wedge d^4y = \int (\lambda J^i \sqrt{|G|})_{;i} d^4x \wedge d^4y - \int \lambda (J^i \sqrt{|G|})_{;i} d^4x \wedge d^4y$$

This term can be written as $\int \operatorname{div}(\lambda J^H) d\Omega - \int \lambda \operatorname{div}(J^H) d\Omega$. According to the continuity equation and taking into account that λ does not depend on y , we can write it as $\int \operatorname{div}(\lambda J^H) d\Omega + \int \operatorname{div}(\lambda J^V) d\Omega = \int \operatorname{div}(\lambda J) d\Omega$, i.e., it can be written as a boundary term. When performing variations of the action (and assuming, as in the classical case, that variations vanish on the boundary), these terms will cancel out.

In conclusion, transformations $A(x, y) \mapsto A(x, y) + d\lambda(x)$ of the 4-potential do not affect the action (9).

Remark. If $A_i = A_i(x)$, then from (52), it follows $J^i = 0$.

8 Equations of motion

Let us consider momentarily the case of a single particle. The equations of motion are obtained by varying the trajectory $x = x(t)$ in the first two terms of (9), which are in our case written in the form of a single integral along the considered curve:

$$S_2 = - \int (mc \sqrt{g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j} + \frac{q}{c} A_k(x, \dot{x}) \dot{x}^k) dt. \quad (55)$$

The 0-homogeneity of A insures that the action S_2 is invariant under eventual changes of parameter $t \mapsto t'$ of the curve.

A further restriction can be imposed on the y -dependence of A in order to make all the approach more elegant and provide a simple relation of A with the canonical 4-momentum and the equations of motion of charged particles.

Once the independence of the integral on the parametrization was established, we are free to choose the parameter along the considered curves. Traditionally, when deducing the equations of motion, curves are parametrized by the arclength s . In this case, the action S_2 in (55) is equivalent to the one provided by the Lagrangian

$$L = \frac{1}{2}mcg_{ij}(x, y)y^i y^j + \frac{q}{c}A_k(x, y)y^k, \quad y = \frac{dx}{ds}, \quad (56)$$

which is more comfortable in view of Legendre duality and Hamiltonian formalism.

The canonical momentum of L is given by

$$p_i = \frac{\partial L}{\partial y^i} = mcy_i + \frac{q}{c}(A_{k \cdot i}y^k + A_i).$$

In *isotropic* (pseudo-Riemannian) spaces, if we assume that $A = A(x)$, then there exists only one potential providing a given interaction Lagrangian $L_{int} = A_i(x)y^i$. But in Finsler spaces, where $A_i = A_i(x, y)$, a given Lagrangian $L_{int} = A_i(x, y)y^i$ can be given by infinitely many functions $A_i = A_i(x, y)$. Thus, to a Lagrangian L_{int} , it corresponds a whole equivalence class of potentials A . Comparing to (19), it appears as convenient to choose from each class the representative for which

$$A_{k \cdot i}y^k = 0. \quad (57)$$

We will call this condition upon A , the *gradient gauge*. In the gradient gauge,

$$A_i = \frac{\partial(A_k y^k)}{\partial y^i}.$$

Remark. 0-homogeneity of A insures that we also have $A_{i \cdot k}y^k = 0$.

In the gradient gauge, the canonical 4-momentum is given by

$$p_i = \frac{\partial L}{\partial y^i} = mcy_i + \frac{q}{c}A_i,$$

in other words, the *Liouville (canonical) 1-form* $\theta = \frac{\partial L}{\partial y^i}dx^i$ attached to L is given by $\theta = (mcy_i + \frac{q}{c}A_i)dx^i$ and the *Poincaré 2-form* $\omega = d\theta$, by

$$\omega = \frac{1}{2}(A_{j|i} - A_{i|j})dx^i \wedge dx^j - (mcg_{ij} + \frac{q}{c}A_{i \cdot j})dx^i \wedge \delta y^{\bar{j}}$$

In the following, we assume that the tangent bundle TM is endowed with an (arbitrary) Ehresmann connection N and the corresponding linear connection $D\Gamma(N)$, (34).

Variation of (56) provides the Euler-Lagrange equations:

$$mc \frac{Dy^i}{ds} = \frac{q}{c} F^i_{\bar{j}} y^j + \frac{q}{c} F^i_{\bar{j}} \frac{\delta y^{\bar{j}}}{ds}, \quad y^i = \frac{dx^i}{ds}, \quad (58)$$

where $\frac{Dy^i}{ds} = \frac{dy^i}{ds} + L^i_{jk} y^j y^k$ and we assumed $A_{k,i} y^k = 0$.

The first term in the right hand side above is similar to the usual one in pseudo-Riemannian spaces, while the second one $\frac{q}{c} F^i_{\bar{j}} \frac{\delta y^{\bar{j}}}{ds}$ is new and appears due to the dependence of A on the variable y .

Remark. Both the "traditional" Lorentz force term (given by $F^i = \frac{q}{c} F^i_h y^h$) and the correction given by $\tilde{F}^i = \frac{q}{c} F^i_{\bar{j}} \frac{\delta y^{\bar{j}}}{ds}$ are orthogonal to the velocity 4-vector $y = \dot{x}$:

$$g_{ij} F^i y^j = 0, \quad g_{ij} \tilde{F}^i y^j = 0. \quad (59)$$

Remark. In equations (58), we can use any Ehresmann connection N , their form does not depend on N .

9 Stress-energy-momentum tensor

9.1 In flat pseudo-Finsler spaces

Let us consider the vector space $M = \mathbb{R}^4$ endowed with a flat pseudo-Finsler metric

$$g_{ij} = g_{ij}(y).$$

Assuming that coordinate transformations are linear (as traditionally done in special relativity), we can choose the trivial Ehresmann connection $N^{\bar{i}}_{\bar{j}} = 0$, hence the adapted frame on TM is the natural one ($\delta_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^{\bar{i}}}$) and its dual is ($dx^i, \delta y^{\bar{i}} = dy^{\bar{i}}$).

Spacetime translations $\bar{x}^i = x^i + \varepsilon^i$, $i = \overline{0,3}$ induce the following transformation on TM :

$$\bar{x}^i = x^i + \varepsilon^i, \quad \bar{y}^i = y^i. \quad (60)$$

By *generalized energy-momentum tensor on TM* , we understand the Noether current given by the invariance to spacetime translations (accordingly, to transformations (60)) of the action

$$S_F = - \int \frac{1}{16\pi c} F * F d^4 x \wedge d^4 y, \quad (61)$$

symmetrized by adding a divergence term.

As pointed out in Section 2, invariance with respect to transformations (60) of the above means the absence of explicit dependence on the base coordinates x^i of the Lagrangian.

In order to find the form of the energy-momentum tensor of the electromagnetic field in a flat pseudo-Finslerian space, let us first see how Noether theorem is read in these spaces.

For an action

$$S = \frac{1}{c} \int \Lambda(q_{(l)}, \frac{\partial q_{(l)}}{\partial x^i}, \frac{\partial q_{(l)}}{\partial y^i}) d\Omega, \quad (62)$$

where $\Lambda = L\sqrt{|G|}$ is a Lagrangian density on TM and $q_{(l)} = q_{(l)}(x, y)$ are the field variables, the Euler-Lagrange equations are:

$$\frac{\partial}{\partial x^i} \left(\frac{\partial \Lambda}{\partial q_{(l),i}} \right) + \frac{\partial}{\partial y^i} \left(\frac{\partial \Lambda}{\partial q_{(l),i}} \right) - \frac{\partial \Lambda}{\partial q_{(l)}} = 0. \quad (63)$$

The absence of explicit dependence on x^i of Λ means

$$\frac{\partial \Lambda}{\partial x^i} = \frac{\partial \Lambda}{\partial q_{(l)}} \frac{\partial q_{(l)}}{\partial x^i} + \frac{\partial \Lambda}{\partial q_{(l),k}} q_{(l),ki} + \frac{\partial \Lambda}{\partial q_{(l),\bar{k}}} q_{(l),\bar{k},i}.$$

(where we understood also summation over l). Substituting $\frac{\partial \Lambda}{\partial q_{(l)}}$ from (63) and grouping terms, we get

$$\frac{\partial}{\partial x^k} \left(q_{(l),i} \frac{\partial \Lambda}{\partial q_{(l),k}} - \delta_i^k \Lambda \right) + \frac{\partial}{\partial y^{\bar{k}}} \left(q_{(l),i} \frac{\partial \Lambda}{\partial q_{(l),\bar{k}}} \right) = 0.$$

Thus, the invariance of an action on TM under translations on the (flat) base space M leads to the appearance of a quantity consisting of *two blocks*, namely, by symmetrizing (adding divergence terms to) the quantities

$$\tilde{T}_i^k = \frac{1}{\sqrt{|G|}} q_{(l),i} \frac{\partial \Lambda}{\partial q_{(l),k}} - \delta_i^k \Lambda, \quad \tilde{T}_i^{\bar{k}} = \frac{1}{\sqrt{|G|}} q_{(l),i} \frac{\partial \Lambda}{\partial q_{(l),\bar{k}}} \quad (64)$$

With these notations,

$$\frac{\partial}{\partial x^k} \left(\tilde{T}_i^k \sqrt{|G|} \right) + \frac{\partial}{\partial y^{\bar{k}}} \left(\tilde{T}_i^{\bar{k}} \sqrt{|G|} \right) = 0.$$

Case 1 ($J = 0$). In order to "guess" the form of the generalized energy-momentum tensor for the electromagnetic field, it is advantageous to assume for the beginning that $J = 0$ and apply the above construction to the Lagrangian density:

$$\Lambda = -\frac{1}{16\pi} F * F = -\frac{1}{16\pi} F_{BC} F^{BC} \sqrt{|G|}.$$

We get $\frac{\partial \Lambda}{\partial A_{k,l}} = -\frac{1}{4\pi} F^{lk} \sqrt{|G|}$, $\frac{\partial \Lambda}{\partial A_{k,\bar{l}}} = -\frac{1}{4\pi} F^{\bar{l}k} \sqrt{|G|}$ and

$$\tilde{T}_i^l = \frac{1}{4\pi} (-F^{lk} A_{k,i} + \frac{1}{4} \delta_i^l F_{BC} F^{BC}), \tilde{T}_i^{\bar{l}} = -\frac{1}{4\pi} F^{\bar{l}k} A_{k,i}.$$

The obtained Noether current can be symmetrized by adding divergence terms. It can be easily seen that, for $J = 0$, the term $\frac{1}{4\pi} (F^{lk} A_{i,k} + F^{\bar{l}k} A_{i,\bar{k}})$ can be expressed as a divergence term.

By adding this term to \tilde{T}_i^l , we get $T_i^l = -\frac{1}{4\pi} (F^{lk} F_{ik} + F^{\bar{l}k} F_{i\bar{k}} - \delta_i^l F_{BC} F^{BC})$ or, equivalently,

$$T_i^l = \frac{1}{4\pi} (-F^{lB} F_{iB} + \frac{1}{4} \delta_i^l F_{BC} F^{BC}). \quad (65)$$

Similarly, $\frac{1}{4\pi} F^{\bar{l}k} A_{i,k}$ can be expressed as a divergence term; by adding it to $\tilde{T}_i^{\bar{l}} = -\frac{1}{4\pi} F^{\bar{l}k} A_{k,i}$, we obtain

$$T_i^{\bar{l}} = -\frac{1}{4\pi} F^{\bar{l}k} F_{ik}. \quad (66)$$

Thus, in the case $J = 0$, we get $\text{div}(T) = 0$, i.e.,

$$\frac{\partial}{\partial x^k} (T_i^k \sqrt{|G|}) + \frac{\partial}{\partial y^{\bar{k}}} (T_i^{\bar{k}} \sqrt{|G|}) = 0.$$

This suggests the following

Definition 3 *The generalized energy-momentum tensor in the flat Finsler space $(\mathbb{R}^4, \mathcal{F}(y))$ is the symmetric tensor*

$$T = T_{ij} dx^i \otimes dx^j + T_{i\bar{j}} dx^i \otimes dy^{\bar{j}} \quad (67)$$

with local components given by (65) and (66).

The horizontal component $T_{ij} dx^i \otimes dx^j$ is the usual energy-momentum tensor (plus some correction due to anisotropy), while the mixed one $T_{i\bar{j}} dx^i \otimes dy^{\bar{j}}$ is new. As we have seen above, these new components play a role in the analogue of the conservation law.

Case 2 ($J \neq 0$). Let now the TM -current J be arbitrary. By using Maxwell equations (with $N = 0$), we get:

$$\frac{1}{\sqrt{|G|}} \left[\frac{\partial}{\partial x^j} (T_i^j \sqrt{|G|}) + \frac{\partial}{\partial y^{\bar{j}}} (T_i^{\bar{j}} \sqrt{|G|}) \right] = -\frac{1}{c} (F_{ij} J^j + F_{i\bar{j}} J^{\bar{j}}). \quad (68)$$

In brief,

$$\text{div}(T) = -\frac{1}{c} i_J F. \quad (69)$$

9.2 In general Finsler spaces

In general (pseudo-)Finsler spaces, we will still define the generalized energy-momentum tensor for the electromagnetic field as above:

$$\begin{aligned} T &= T_{ij}dx^i \otimes dx^j + T_{i\bar{j}}dx^i \otimes \delta y^{\bar{j}}, \\ T_{iA} &= \frac{1}{4\pi}(-F_A{}^B F_{iB} + \frac{1}{4}g_{iA}F_{BC}F^{BC}), \end{aligned} \quad (70)$$

where $g_{i\bar{j}} = 0$ and indices A, B, C take all values corresponding to both horizontal and vertical components.

Remark. The horizontal components T_{ij} of the generalized energy-momentum tensor can be obtained by varying the action S_F with respect to the spacetime metric g (i.e., with respect to the *horizontal* part of the metric (G_{AB})):

$$\delta_g S_F = \frac{1}{2c} \int T_{ik} \delta g^{ik} d\Omega,$$

while the mixed components $T_{i\bar{j}}$ are obtained by varying (independently) S_F with respect to the Ehresmann connection N :

$$\delta_N S_F = \frac{1}{c} \int T_{i\bar{j}}^j \delta N_{\bar{j}}^i d\Omega.$$

In curved pseudo-Finsler spaces case, the adapted frame $(\delta_i, \partial_{\bar{i}})$ is generally nonholonomic, hence any linear connection $D\Gamma(N)$ which preserves the distributions generated by N has generally nonvanishing torsion (at least, $\delta y^{\bar{i}}(T(\delta_k, \delta_j)) = R^{\bar{i}}{}_{jk} \neq 0$). In this case, the covariant divergence of the energy-momentum tensor is not simply equal to $-\frac{1}{c}i_J F$, but has a more complicated expression, involving the torsion tensor. The situation formally resembles to the one in Riemann-Cartan geometry, [32].

In order to find the relation between $\frac{1}{c}i_J F$ and the generalized energy-momentum tensor, it appears as most comfortable to express $-\frac{1}{c}i_J F = -\frac{1}{c}(F_{ij}J^j + F_{i\bar{j}}J^{\bar{j}})$ in terms of covariant derivatives (34).

Let us assume for simplicity that $v = v(x)$ and N is given by (31). Then,

$$Q^i = F^{i\bar{j}}C_{h\bar{j}}^h, \quad Q^{\bar{i}} = -\frac{1}{2}F^{jk}R^{\bar{i}}{}_{jk}. \quad (71)$$

Taking into account the Maxwell equations, we get

$$-\frac{1}{c}(F_{ij}J^j + F_{i\bar{j}}J^{\bar{j}}) = T_{i|j}^j + T_{i\bar{j}}^{\bar{j}} + T_{i\bar{j}}^{\bar{j}}C_{h\bar{j}}^h + T_{\bar{k}}^j R^{\bar{k}}{}_{ij}. \quad (72)$$

10 Conclusion

For a 4-dimensional pseudo-Finsler space (M, \mathcal{F}) , we have constructed a notion of electromagnetic tensor, based exclusively on variational calculus and exterior derivative, [30].

The 4-potential is defined as a horizontal 1-form $A = A_i(x, y)dx^i$ on the tangent bundle TM , having its components A_i homogeneous of degree 0 in y .

In terms of this 4-potential, the generalized electromagnetic tensor is the 2-form $F = dA$. The Maxwell equations on TM are then written as:

$$dF = 0, \quad \delta F = -\frac{4\pi}{c}J_b.$$

The TM -current $J = J^i\delta_i + J^{\bar{i}}\partial_{\bar{i}}$ is a vector field on TM satisfying identically $\text{div } J = 0$. Its horizontal component $J^i\delta_i$ provides the usual notion of 4-current (plus a correction term due to the y -dependence of A), while the vertical one $J^{\bar{i}}\partial_{\bar{i}}$ is due to the anisotropy of the 4-potential and to the nonholonomy of the frame.

Further, for flat pseudo-Finsler spaces $(M, \mathcal{F}(y))$, the generalized energy-momentum tensor is defined as the symmetrized Noether current corresponding to invariance to spacetime translations of the field Lagrangian. We obtained

$$T = T_{ij}dx^i \otimes dx^j + T_{i\bar{j}}dx^i \otimes dy^{\bar{j}}, \quad (73)$$

$$T_{iA} = \frac{1}{4\pi}(-F_A{}^B F_{iB} + \frac{1}{4}\delta_A^l F_{BC} F^{BC}), \quad (74)$$

(where δ_j^l is the Kronecker delta and $\delta_j^l = 0$ and A, B, C take all values corresponding to both horizontal and vertical components). The generalized energy-momentum tensor satisfies the conservation law

$$\text{div}(T) = -\frac{1}{c}(F_{ij}J^j + F_{i\bar{j}}J^{\bar{j}}).$$

In curved Finsler spaces, the same expressions can be obtained by varying by varying the action $S_F = -\int \frac{1}{16\pi}F * F d\Omega$ for the field with respect to the metric $g_{ij}(x, y)$ (thus getting T_{ij}) and with respect to the Ehresmann connection N (which provides the components, $T_{i\bar{j}}$).

The above considerations hold true in more general spaces such as Lagrange or generalized Lagrange spaces, with the only mention that in these spaces, the equations of motion of charged particles become more complicated (and the 0-homogeneity assumption on A is dropped).

Remark. Prior to this model, to our knowledge, there existed only one geometric model for electromagnetism in Finsler spaces, belonging to R. Miron and collaborators, [15], [18], [17], [19], in which it is defined, by means of deflection tensors of linear connections on TM , a notion of electromagnetic tensor on TM (with horizontal hh - and vertical vv - components) and it is provided a generalization of Maxwell equations. Several of the the advantages of our approach are:

obtaining by variational methods the TM -versions of: Maxwell equations, equations of motion (and Lorentz force, respectively), energy-momentum tensor; an easier interpretation of the new (mixed hv -) component of the electromagnetic 2-form (as appearing, for instance, in the equations of motion); obtaining an analogue of the usual continuity equation as an identity, and also, the possibility of a compact writing using exterior derivatives.

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